

DISCRETE ALGEBRAIC EQUATIONS AND DISCRETE OPERATOR EQUATIONS(PRESENTATIONS FOR ICM 2010)

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ABSTRACT. We give constructive results of Hilbert's 13th problem for discrete functions. By them we give formula solution expressed by a superposition of functions of one variable to equations constructed by discrete functions and equations with parameterized discrete functions. Further more we give formula solution expressed by a superposition of operators of one variable to equations constructed by discrete operators and equations with parameterized discrete operators. This is a Short communication, Section 9, Functional Analysis and Application, Saturday, August 21, 2010, 18:00-18:15, Room No. T3.

1. INTRODUCTION

Problems about equations are very important and difficult. Solving quadratic equation and cubic equation and quartic equation had cost the mathematicians in history a great deal of time.

Babylonians solve quadratics in radicals in 2000 BC. Cubic equation and quartic equation were solved by Italian mathematicians Girolamo Cardano (1501-1576) and Ludovico Ferrari (1522-1565) in 16th century, respectively.

But mathematicians met big troubles when they tried to solve quintic equation. Leonhard Euler (1707-1783) believed quintic equation can be changed to a quartic equation by transformation of variable. Niels Henrik Abel (1802-1829) got a conclusion that there is no solution by radicals for a general polynomial algebraic equation if $n \geq 5$. Evariste Galois (1811-1832) built group theory and got the same conclusion. His method come down to now and can be found in any textbook about Galois group theory.

There is no solution by radicals. Are there any solutions of other forms such as numerical solution and solution expressed in function of two variables or of many variables or solution expressed in series or in integral?

We do not discuss numerical solutions because they belong to applied mathematics. We prefer formula solution expressed in binary function to other ones. What is a formula solution expressed in binary function? It contains only function of two variables. We can give an expression of a alone binary function at the beginning. We can replace any one of variables by a binary function then we get a new expression. We can replace any one of variables of this new expression by a binary function again and get a more complex expression. We can repeat the procedure for any finite times. But it is not easy to get solution expressed in binary function. It's easier to get solutions of other forms. History developed just like this.

Key words and phrases. Discrete function, commutation operator, tension-compression operator, superposition operator, decomposition operator, discrete operator, high operator.

Camille.Jordan (1838-1922) shows that algebraic equations of any degree can be solved in terms of modular functions in 1870. Ferdinand.von.Lindemann (1852-1939) expresses the roots of an arbitrary polynomial in terms of theta functions in 1892. In 1895 Emory.McClintock (1840-1916) gives series solutions for all the roots of a polynomial. Robert Hjalmar.Mellin (1854-1933) solves an arbitrary polynomial equation with Mellin integrals in 1915. In 1925 R.Birkeland shows that the roots of an algebraic equation can be expressed using hypergeometric functions in several variables. Hiroshi.Umemura expresses the roots of an arbitrary polynomial through elliptic Siegel functions in 1984[1].

All of solutions mentioned above are not ones expressed in binary function.

By Tschirnhausen transformation a quintic equation or a sextic equation can be changed to ones containing only two parameters so there are solutions expressed in binary function for them. David.Hilbert presumed that there is no solution expressed in binary function for polynomial equations of n when $n \geq 7$ and wrote his doubt into his famous 23 problems as the 13th one[2].

Hilbert published his last mathematical paper [3] in 1927 where he reported on the status of his problems, he devoted 5 pages to the 13th problem and only 3 pages to the remaining 22 problems. We can see that so much attention Hilbert paid to 13th problem. In 1957 V.I.Arnol'd proved that every continuing function of many variables can be represented as a superposition of functions of two variables and refuted Hilbert conjecture[4][5]. Furthermore, A.N.Kolmogorov proved that every continuous function of several variables can be represented as a superposition of continuous functions of one variable and the operation of addition [6].

Result for Hilbert's 13th problem is very important for us and it points us a quite right direction to solve polynomial equations and general algebraic equations. But method used in it is topological and the result is not a constructive one. In this paper we will give a constructive result in discrete situation. This result is very important. We can construct profuse discrete algebraic equations and discrete operator equations and for this result we can give any of them a formula solution. There is never such a mathematical structure in the history of mathematics. This is the first time! A.G.Vitushkin dissatisfies the current results about 13th problem and points out that the algebraic core of the problem remains untouched[7]. We believe we have gotten the algebraic core Vitushkin wanted.

2. CONSTRUCTIVE RESULTS FOR HILBERT'S 13TH PROBLEM

A.N.Kolmogorov expresses function of several variables as a superposition of functions of one variable like this:

$$(2.1) \quad W(x_1, x_2, \dots, x_n) = \sum_{i=1}^{2n+1} f_i \left[g_{i1}(x_1) + g_{i2}(x_2) + \dots + g_{in}(x_n) \right]$$

This is a existence result but it's easy to give a constructive result for discrete functions.

Definition 2.1 Let $A = \{-1, 0, 1\}$, a three numbers function of M variables is defined as:

$$g: A^M \longrightarrow A$$

There are 9 discrete points for a binary three numbers function. A binary three numbers function can be indicated by a table with 4x4 elements. Its first column indicates the first variable and the first row indicates the second variable. To give

a table with 4x4 elements is to define a binary three numbers function and vice versa. For example:

	-1	0	1
-1	1	-1	0
0	-1	0	1
1	0	1	-1

	-1	0	1
-1	0	0	0
0	0	0	0
1	0	0	0

	-1	0	1
-1	1	1	1
0	0	0	0
1	-1	-1	-1

There are three functions in above tables. The first one is linear binary three numbers function and the second one is identity function with 0 value and value of the third one is not change with the second variable. There is only one value for each discrete point in these three functions and they are called single-valued binary three numbers function. It's easy to know there are $3^9=19683$ single-valued binary three numbers function. Can it be two-valued or three-valued in a discrete point? certainly! There are three combinations -1,0 and -1,1 and 0,1 for two-valued and only one combination -1,0,1 for three-valued and numbers will be partitioned by symbol '*' if it's a multi-valued. Can it be no-valued in a discrete point? Yes! We will indicate it in 'N'. A binary three numbers function can be no-valued in all 9 discrete points in uttermost like:

	-1	0	1
-1	N	N	N
0	N	N	N
1	N	N	N

There is single-valued,two-valued,three-valued and no valued point in below three numbers function.

	-1	0	1
-1	-1	0	1
0	-1*0	-1*1	0*1
1	N	-1*0*1	N

It's easy to know there are 8^9 binary three numbers functions. A unary three numbers function can be indicated by three value numbers partitioned by symbol ',' in bracket and numbers will be partitioned by the symbol '*' if it's many-valued, for example:(-1*0,N,-1*0*1).

The expression $H(x_1, x_2) = f[g_{11}(x_1) + g_{12}(x_2)]$ contains x_1, x_2 , but we intend to take H as a independent object not containing x_1, x_2 . We can't express H in $f[g_{11} + g_{12}]$ because we will get a unary function $[g_{11} + g_{12}]$ by adding g_{11} and g_{12} . $f[g_{11} + g_{12}]$ is also a unary function and is never equal to binary the function H.

We express H by only f, g_{11} , g_{12} without x_1, x_2 like this:

$$H = f[g_{11}\widetilde{\alpha}_1 + g_{12}\widetilde{\alpha}_2]$$

To define a function is to give a rule to get it's values. For such an expression we are very clear the rule about getting values of the function if we replace $\widetilde{\alpha}_1$ or $\widetilde{\alpha}_2$ by x_1 or x_2 , respectively. That is enough.

Binary three numbers function is called single term binary three numbers function if it can be represented as $H = f[g_{11}\widetilde{\alpha}_1 + g_{12}\widetilde{\alpha}_2]$ in which f, g_{ij} is unary three numbers function and it will be called L term binary three numbers function if it can be expressed as $\sum f_i[g_{i1}\widetilde{\alpha}_1 + g_{i2}\widetilde{\alpha}_2](i = 1, L)$. Expressing a function of many variables as this form is also called representing it as a superposition of functions of one variable or decomposing it to functions of one variable.

For example F is a single term binary three numbers function:

$$F = (0, 0, 1) \left[(0, 0, 1)\widetilde{\alpha}_1 + (-1, -1, 0)\widetilde{\alpha}_2 \right]$$

Theorem 2.1 Every binary three numbers function can be represented as a superposition of three numbers functions of one variable.

A binary three numbers function is called singular binary three numbers function if it's zero in all discrete points but except one. It's called standard singular three numbers binary function if non-zero point is in lower-right location. Definitions for singular three numbers function of three variables and for standard singular three numbers function of three variables are similar.

First we prove that the standard singular three numbers binary function is a single term one. It's clear the standard singular binary three numbers function is F above. $(0,0,1)$ in $(0,0,1)\widetilde{\alpha}_1$ in the expression of F is called raw function. Raw of none-zero point will change if we adjust the location of '1' in $(0,0,1)$. $(-1,-1,0)$ of $(-1,-1,0)\widetilde{\alpha}_2$ in it is called column function. Column of none-zero point will change if we adjust the location of '0' in $(-1,-1,0)$. The first $(0,0,1)$ in it is called value function. Value which may be single-valued or multi-valued or no-valued of non-zero point will change if we modify '1' in $(0,0,1)$. Thus we know that every singular binary three numbers function can be represented as a superposition of three numbers functions of one variable.

Because every binary three numbers function can be transformed to sum of 9 singular binary three numbers functions then we get our theorem.

So every binary three numbers function can be represented as:

$$(2.2) \quad \Psi_2 = \sum_{i=1}^L f_i [g_{i1}\widetilde{\alpha}_1 + g_{i2}\widetilde{\alpha}_2]$$

Here L is not greater than 9. Thus we can express and can construct a binary three numbers function by unary three numbers functions.

We can extend all these result to N numbers function of several variables. In the decomposition of standard singular binary three numbers function if we replace raw function $(0,0,1)$ by $(0,0,\dots,0,1)$, column function $(-1,-1,0)$ by $(-1,-1,\dots,-1,0)$ and value function $(0,0,1)$ by $(0,0,\dots,0,1)$, respectively. Then we can extend this expression to N numbers functions of two variables. Situation for N numbers functions of M variables is similar. So we get conclusion below.

If $N \geq M+1$ a general N numbers function of M variables can be decomposed as:

$$(2.3) \quad \psi = \sum_{i=1}^L f_i \sum_{j=1}^M g_{ij}\widetilde{\alpha}_i$$

If $N < M+1$, the number of unary function in expression of singular discrete function will be bigger than M+1. For example a standard singular three numbers function of three variables 3 can be represented as:

$$(2.4) \quad \Psi_3 = (0,0,1)\{(0,0,1)[(0,0,1)\widetilde{\alpha}_1 + (-1,-1,0)\widetilde{\alpha}_2] + (-1,-1,0)\widetilde{\alpha}_3\}$$

Here are more location functions $(-1,-1,0)$ and $(0,0,1)$ than one of the standard singular binary three numbers function. Expressions for singular three numbers function of three variables and for general three numbers function of three variables are similar to ones of binary three numbers functions.

All conclusions here are not suit to two numbers function. So we have:

Theorem 2.2 Every N numbers ($N \geq 3$) function of M variables can be represented as a superposition of N numbers functions of one variable.

Note we not only prove the existence of representation by superposition of functions of one variable and give a constructive procedure. We just only gave the method to decomposing a function but expression is not the shortest one. Decomposition with terms being equal to its discrete points is called a trivial decomposition. Actual terms are more less. Decomposition with less terms than trivial decomposition is called non-trivial decomposition. It's an important topic to study non-trivial decompositions and will not be stated here.

3. EQUATIONS CONSTRUCTED BY THREE NUMBERS FUNCTIONS

There are 3^9 single-valued binary three numbers function and 8^9 ones if they contain many-valued or no-valued ones. How many equations can we construct with these functions? So many! How many things need to study about group of the order 3? Too poor! So we know there are ample mineral resources in this task.

Theorem 3.1 Every algebraic equation constructed by three numbers functions of two variables can be represented as a superposition of three numbers functions of one variable.

It's simple to improve it. Solution of any equation is always function of several variables. By substituting -1,0,1 to the equation respectively we can get this function easily because field of definition of it is only three numbers -1,0,1. We can get the solution expressed by function of one variable by decomposing this function. That is wonderful that we can construct equations and solve them freely in a mathematics system! In this paper we just only solve the equation though there are multitudinous equations:

$$(x\psi_1a)\psi_3(x\psi_2b) = c$$

Here and in this paper we do't write functions of two variables in prefix form like $\psi_3[\psi_1(x,a), \psi_2(x,b)] = c$ for clearness. This equation is called two branches equation. ψ_i is parameterized function and can be any one of 8^9 three numbers functions of two variables. So actually we solve not one equation but a kind of equation and the method possesses universality.

Assume function ψ_1, ψ_2 and ψ_3 in the two branches equation is Ω_1, Ω_2 and Ω_3 , respectively:

	-1	0	1
-1	-1	1	0
0	0	-1	1
1	1	0	-1

	-1	0	1
-1	0	-1	1
0	-1	0	-1
1	1	1	0

	-1	0	1
-1	1	-1	0
0	0	1	-1
1	-1	0	1

When $a=b=c=-1$ we get numerical equation $[x\Omega_1(-1)]\Omega_3[x\Omega_2(-1)] = -1$. We know only -1 is the solution of this equation by substituting -1,0,1 to it. So we can know that $W(-1,-1,-1)=-1$. By the same way we can get other values of $W(a,b,c)$. $W(a,b,c)$ can be expressed by table below.

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	-1	N	N		0	N	-1*0*1		1	-1*0*1	N
0	1	-1*0*1	N		-1	N	N		0	N	-1*0*1
1	0	N	-1*0*1		1	-1*0*1	N		-1	N	N

In this table the first column indicates the first function number a and the first row indicates the second function number b and c indicates the third function number. Decomposing this function of three variables we get the solution expressed by a superposition of functions of one variable.

$$\begin{aligned}
x = & (0, 0, -1) \left\{ (0, 0, 1) \left[(1, 0, 0)a + (0, -1, -1)b \right] + (0, -1, -1)c \right\} \\
& + (0, 0, 1) \left\{ (0, 0, 1) \left[(1, 0, 0)a + (0, -1, -1)b \right] + (-1, -1, 0)c \right\} \\
& + (0, 0, N) \left\{ (0, 0, 1) \left[(1, 0, 0)a + (-1, 0, -1)b \right] + (0, -1, -1)c \right\} \\
& + (0, 0, N) \left\{ (0, 0, 1) \left[(1, 0, 0)a + (-1, 0, -1)b \right] + (-1, 0, -1)c \right\} \\
& + (0, 0, -1*0*1) \left\{ (0, 0, 1) \left[(1, 0, 0)a + (-1, 0, -1)b \right] + (-1, -1, 0)c \right\} \\
& + (0, 0, N) \left\{ (0, 0, 1) \left[(1, 0, 0)a + (-1, -1, 0)b \right] + (0, -1, -1)c \right\} \\
& + (0, 0, -1*0*1) \left\{ (0, 0, 1) \left[(1, 0, 0)a + (-1, -1, 0)b \right] + (-1, 0, -1)c \right\} \\
& + (0, 0, N) \left\{ (0, 0, 1) \left[(1, 0, 0)a + (-1, -1, 0)b \right] + (-1, -1, 0)c \right\} \\
& + (0, 0, 1) \left\{ (0, 0, 1) \left[(0, 1, 0)a + (0, -1, -1)b \right] + (0, -1, -1)c \right\} \\
& + (0, 0, -1) \left\{ (0, 0, 1) \left[(0, 1, 0)a + (0, -1, -1)b \right] + (-1, 0, -1)c \right\} \\
& + (0, 0, -1*0*1) \left\{ (0, 0, 1) \left[(0, 1, 0)a + (-1, 0, -1)b \right] + (0, -1, -1)c \right\} \\
& + (0, 0, N) \left\{ (0, 0, 1) \left[(0, 1, 0)a + (-1, 0, -1)b \right] + (-1, 0, -1)c \right\} \\
& + (0, 0, N) \left\{ (0, 0, 1) \left[(0, 1, 0)a + (-1, 0, -1)b \right] + (-1, -1, 0)c \right\} \\
& + (0, 0, N) \left\{ (0, 0, 1) \left[(0, 1, 0)a + (-1, -1, 0)b \right] + (0, -1, -1)c \right\} \\
& + (0, 0, N) \left\{ (0, 0, 1) \left[(0, 1, 0)a + (-1, -1, 0)b \right] + (-1, 0, -1)c \right\} \\
& + (0, 0, -1*0*1) \left\{ (0, 0, 1) \left[(0, 1, 0)a + (-1, -1, 0)b \right] + (-1, -1, 0)c \right\} \\
& + (0, 0, 1) \left\{ (0, 0, 1) \left[(0, 0, 1)a + (0, -1, -1)b \right] + (-1, 0, -1)c \right\} \\
& + (0, 0, -1) \left\{ (0, 0, 1) \left[(0, 0, 1)a + (0, -1, -1)b \right] + (-1, -1, 0)c \right\}
\end{aligned}$$

$$\begin{aligned}
 & +(0,0,N)\left\{(0,0,1)\left[(0,0,1)a+(-1,0,-1)b\right]+(0,-1,-1)c\right\} \\
 & +(0,0,-1*0*1)\left\{(0,0,1)\left[(0,0,1)a+(-1,0,-1)b\right]+(-1,0,-1)c\right\} \\
 & +(0,0,N)\left\{(0,0,1)\left[(0,0,1)a+(-1,0,-1)b\right]+(-1,-1,0)c\right\} \\
 & +(0,0,-1*0*1)\left\{(0,0,1)\left[(0,0,1)a+(-1,-1,0)b\right]+(0,-1,-1)c\right\} \\
 & +(0,0,N)\left\{(0,0,1)\left[(0,0,1)a+(-1,-1,0)b\right]+(-1,0,-1)c\right\} \\
 & +(0,0,N)\left\{(0,0,1)\left[(0,0,1)a+(-1,-1,0)b\right]+(-1,-1,0)c\right\}
 \end{aligned}$$

There are 24 but not 27 terms because there are three discrete point with 0 value.

4. FOUR SPECIAL OPERATORS

How to get a new function from a known one? Operator is correspondence between functions. To give a correspondence between known functions and new functions is to give an operator. Four special operators mentioned here are easy to be understood intuitively and are important to solve equations with parameterized functions however so we must pay attention to them.

Definition 4.1Commutation operators. Assume there is an function of two variables ψ , $a_1\psi a_2 = a_0$, its commutation functions $\psi(1,2,0), \psi(1,0,2), \psi(0,2,1), \psi(2,1,0), \psi(2,0,1), \psi(0,1,2)$ will be defined by following formulas and we introduce commutation operators of one variable $C[1,2,0]$, $C[1,0,2]$, $C[0,2,1]$, $C[2,1,0]$, $C[2,0,1]$, $C[0,1,2]$ then new functions can be expressed by ψ and commutation operators.

$$(4.1a) \quad a_1\psi[1,2,0]a_2 = a_0 \quad \psi(1,2,0) = C(1,2,0)(\psi)$$

$$(4.1b) \quad a_1\psi[1,0,2]a_0 = a_2 \quad \psi(1,0,2) = C(1,0,2)(\psi)$$

$$(4.1c) \quad a_0\psi[0,2,1]a_2 = a_1 \quad \psi(0,2,1) = C(0,2,1)(\psi)$$

$$(4.1d) \quad a_2\psi[2,1,0]a_1 = a_0 \quad \psi(2,1,0) = C(2,1,0)(\psi)$$

$$(4.1e) \quad a_2\psi[2,0,1]a_0 = a_1 \quad \psi(2,0,1) = C(2,0,1)(\psi)$$

$$(4.1f) \quad a_0\psi[0,1,2]a_1 = a_2 \quad \psi(0,1,2) = C(0,1,2)(\psi)$$

Note $\psi(1,2,0)$ is ψ itself.

Numbers in brackets indicates new locations of function numbers and of function result after commuting. That is say original function doesn't satisfy the new relation gotten by commuting location of function numbers and of function result but new one satisfies it. New relation with new function and new location is equivalent to original one in despite of their forms are different. For example: if Ω is the first table below then $C(1,0,2)(\Omega)$, $C(0,2,1)(\Omega)$, $C(2,1,0)(\Omega)$, $C(2,0,1)(\Omega)$, $C(0,1,2)(\Omega)$ will be other tables, respectively.

	-1	0	1
-1	-1	1	0
0	0	-1	1
1	1	0	-1
	-1	0	1
-1	-1	0	1
0	1	-1	0
1	0	1	-1

	-1	0	1
-1	-1	1	0
0	0	-1	1
1	1	0	-1
	-1	0	1
-1	-1	0	1
0	0	1	-1
1	1	-1	0

	-1	0	1
-1	-1	0	1
0	0	1	-1
1	1	-1	0
	-1	0	1
-1	-1	0	1
0	1	-1	0
1	0	1	-1

We can get any combination of function numbers and of function result for $C(1,0,2)(\Omega)$ by commuting the second function number and function result for $C(1,2,0)(\Omega)$. Situations for other commutation functions are similar to it. We don't limit function at all when we do commutation operator. May be we get a many-valued function by a not monotonic function or get an function with no values in some discrete points by a not surjective function. The same situation may be exists in other three special operators. We have shown our opinion above. An mathematics system is extensive and open if it involves solving equation so it's impossible to limit functions in it. I have ever tried to limit function in ones of single-valued but failed because function of many-valued or of no-valued can be introduced from function of single-valued by special operators. This problem had troubled me for a long time until I read materials about extension of group. I known functions of many-valued or of no-valued are not difficult to be accepted by mathematicians. Commutation operator for binary functions can be extended to function of many variables. Showing all commutation functions is integrity in logical and not all of them will be used in solving equations. There are only two commutation functions for a unary function:

$$(4.2a) \quad \beta_e(a) = a_0 \quad \beta_e = \beta$$

$$(4.2b) \quad \beta_t(a_0) = a \quad \beta_t = C(\beta)$$

Definition 4.2 Tension-compression operator. Assume there is a binary function ψ and an unary function β , $\beta(a_1)\psi a_2 = a_0$, we can introduce a new binary function ψ_1 by ψ and β , ψ_1 will meet the relation: $a_1\psi_1 a_2 = a_0$, that is say, $a_1\psi_1 a_2 = \beta(a_1)\psi a_2$. Introduce a special operator T_1 to express the relation between ψ_1 and ψ, β .

$$(4.3a) \quad \psi_1 = \psi T_1 \beta$$

In the same way if $a_1\psi\beta(a_2) = a_0$, we can introduce a new binary function ψ_2 by ψ and β , ψ_2 will meet the relation: $a_1\psi_2 a_2 = a_0$, that is say, $a_1\psi_2 a_2 = a_1\psi\beta(a_2)$. Introduce a special operator T_2 to express the relation between ψ_2 and ψ, β .

$$(4.3b) \quad \psi_2 = \psi T_2 \beta$$

If $a_1\psi a_2 = \beta(a_0)$, that is say $\beta^{-1}[a_1\psi a_2] = a_0$, we can introduce a new binary function ψ_0 by ψ and β , ψ_0 will meet the relation: $a_1\psi_0 a_2 = a_0$, that is say $a_1\psi_0 a_2 = \beta^{-1}[a_1\psi a_2]$, and there is T_0 :

$$(4.3c) \quad \psi_0 = \psi T_0 \beta$$

For example, $(1,-1,0)$ is an function of one variable and written in γ then $\Omega T_1 \gamma$ and $\Omega T_2 \gamma$ and $\Omega T_0 \gamma$ will be

	-1	0	1
-1	1	0	-1
0	-1	1	0
1	0	-1	1

	-1	0	1
-1	0	-1	1
0	1	0	-1
1	-1	1	0

	-1	0	1
-1	0	-1	1
0	1	0	-1
1	-1	1	0

respec-

tively.

It's occasional that $\Omega T_2 \gamma$ is equal to $\Omega T_0 \gamma$. Only T_0 will be used in solving equation.

For an unary function we have only T and T_0 :

$$(4.4a) \quad \beta_1 T \beta_2 = \beta_1 \beta_2$$

$$(4.4b) \quad \beta_1 T_0 \beta_2 = \beta_2^{-1} \beta_1$$

Note, $\beta_1 \beta_2$ means applying β_2 first and then applying β_1 . That is say

$$(4.5) \quad \beta_1 \beta_2(x) = \beta_1 \left[\beta_2(x) \right]$$

A discrete point for $\beta_1 \beta_2$ will be no-valued if it for any of β_1 or β_2 is no-valued. β_1 and β_2 will be each other inverse function if $\beta_1 \beta_2 = e$. There are 8^3 three numbers functions of one variable in which there is always inverse function for any three numbers function of one variable.

This rule is right for many-valued functions of two variables because tension-compression operators for functions of two variables involves actually only composition of two functions of one variable.

Definition 4.3 Superposition operator. Assume there are P functions of many variables $\psi_k (k=1, p)$, their superposition function ψ will be:

$$(4.6) \quad \psi = \sum_{k=1}^P \psi_k$$

Value of ψ will be the sum of value of $\psi_k (k=1, p)$. This is a kind of operator by it we can get a new function by several known functions with same variables. ψ will be no-valued in a point if any of ψ_k is no-valued in this point. $\psi_1 + \psi_2$ will be many-valued in a point if ψ_1 is single-valued and ψ_2 is many-valued in this point.

Definition 4.4 Decomposition operator.

$$(4.7) \quad \psi_3 = \sum_{i=1}^{27} f_i \left\{ g_{i4} \left[g_{i1}(\widetilde{\alpha_1}) + g_{i2}(\widetilde{\alpha_2}) \right] + g_{i3}(\widetilde{\alpha_3}) \right\}$$

We can express the relations between f_i or g_{ij} and ψ_3 with special operators V_3 and P_{ij} and actually g_{ij} is not change with ψ_3 .

$$(4.8a) \quad f_i = V_i(\psi_3) \quad (i = 1, 27)$$

$$(4.8b) \quad g_{ij} = P_{ij}(\psi_3) \quad (i = 1, 27, j = 1, 4)$$

Otherwise there are more than one decomposition for any function of 3 variables but we select only one of them. Correspondence between ψ_3 and f_i, g_{ij} is clear and easy to be gotten. So decomposition operator is not occult at all.

Please note commutation operator or tension-compression operator or decomposition operator or superposition operator will be close within all three numbers functions if they contain ones being many-valued and no-valued. This is very important and is the sufficient reason for existing of many-valued functions and no-valued functions.

So four special operators are very clear and not perplexed at all.

Definition 4.5 False function of M+K variables. We can change an function of M variables to a false one of (M+K) variables by adding $o\widetilde{\alpha}_k$ in which o is a zero function and function ψ will not change with K variables.

$$(4.9) \quad \psi = \sum_{i=1}^L f_i \sum_{j=1}^M g_{ij} \widetilde{\alpha}_i = \sum_{i=1}^L f_i \left\{ \sum_{j=1}^M g_{ij} \widetilde{\alpha}_i + \sum_{k=M+1}^{M+K} o\widetilde{\alpha}_k \right\}$$

We can also get false function of (M+K) variables from one of M variables by $T_k o$ (k=M, M+K). For example:

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	0	1	-1		0,1	N	-1*0*1		1	0	N
0	0	1	-1		0,1	N	-1*0*1		1	0	N
1	0	1	-1		0,1	N	-1*0*1		1	0	N

This is a false function of three variables and value of it will not change with the first variable. Below table is a false function of two variables.

	-1	0	1
-1	1	1	1
0	-1	-1	-1
1	0	0	0

False function of many variables will be used in solving equations with parameterized functions.

5. FORMULA SOLUTION FOR EQUATIONS WITH PARAMETERIZED FUNCTIONS

What's an analytic solution or formula solution for an equation? Formula solution can only contain known parameters or constants and known parameterized functions or known numerical functions and four kinds of special operators and we call them valid symbols and all others invalid ones. This is the standard to verify a formula solution of an equation. Commutation operators and tension-compression operators and superposition operators and decomposition operators are the sufficient condition but not the necessary condition to give formula solutions of equations. There may be another equivalence set of operators that can express formula solutions of equations. We will solve two branches equation as a example below. At the same time we will solve an equation with digital functions below then we can understand the procedure more clearly. We must believe that it's not complex to solve this equation because we have known already the solution exists surely and only four special operators will be deal with to get it. We will take any new function met in procedure of solving the equation as a normal one and will never be puzzled by its appearance.

Step 1: Decomposing function ψ_3 as:

$$\psi_3 = \sum_{i=1}^9 f_i (g_{i1} \widetilde{\alpha}_1 + g_{i2} \widetilde{\alpha}_2)$$

$$\sum_{i=1}^9 f_i [g_{i1}(x\psi_1 a) + g_{i2}(x\psi_2 b)] = c$$

$$\Omega_3 = (0, 0, 1) [(1, 0, 0) \widetilde{\alpha}_1 + (0, -1, -1) \widetilde{\alpha}_2] + (0, 0, -1) [(1, 0, 0) \widetilde{\alpha}_1 + (-1, 0, -1) \widetilde{\alpha}_2]$$

$$\begin{aligned}
 & + (0, 01) \left[(0, 1, 0) \widetilde{\alpha_1} + (-1, 0, -1) \widetilde{\alpha_2} \right] + (0, 0, -1) \left[(0, 1, 0) \widetilde{\alpha_1} + (-1, -1, 0) \widetilde{\alpha_2} \right] \\
 & + (0, 0, -1) \left[(0, 0, 1) \widetilde{\alpha_1} + (0, -1, -1) \widetilde{\alpha_2} \right] + (0, 0, 1) \left[(0, 0, 1) \widetilde{\alpha_1} + (-1, -1, 0) \widetilde{\alpha_2} \right] \\
 & (x\Omega_1 a) \Omega_3 (x\Omega_2 b) = \\
 & (0, 0, 1) \left[(1, 0, 0)(x\Omega_1 a) + (0, -1, -1)(x\Omega_2 b) \right] + (0, 0, -1) \left[(1, 0, 0)(x\Omega_1 a) + (-1, 0, -1)(x\Omega_2 b) \right] \\
 & + (0, 01) \left[(0, 1, 0)(x\Omega_1 a) + (-1, 0, -1)(x\Omega_2 b) \right] + (0, 0, -1) \left[(0, 1, 0)(x\Omega_1 a) + (-1, -1, 0)(x\Omega_2 b) \right] \\
 & + (0, 0, -1) \left[(0, 0, 1)(x\Omega_1 a) + (0, -1, -1)(x\Omega_2 b) \right] + (0, 0, 1) \left[(0, 0, 1)(x\Omega_1 a) + (-1, -1, 0)(x\Omega_2 b) \right] \\
 & =_c
 \end{aligned}$$

Step 2: By tension-compression of g_{i1}, g_{i2} we have:

$$\sum_{i=1}^9 f_i \left[x(\psi_1 T_0 g_{i1}^{-1}) a + x(\psi_2 T_0 g_{i2}^{-1}) b \right] = c$$

Note, $(\psi_1 T_0 g_{i1}^{-1})$ in $x(\psi_1 T_0 g_{i1}^{-1}) a$ and $(\psi_2 T_0 g_{i2}^{-1})$ in $x(\psi_2 T_0 g_{i2}^{-1}) b$ are two functions of two variables.

$$\begin{aligned}
 (x\Omega_1 a) \Omega_3 (x\Omega_2 b) = & (0, 0, 1) \left\{ x \left[\Omega_1 T_0(1, 0, 0)^{-1} \right] a + x \left[\Omega_2 T_0(0, -1, -1)^{-1} \right] b \right\} + \\
 & (0, 0, -1) \left\{ x \left[\Omega_1 T_0(1, 0, 0)^{-1} \right] a + x \left[\Omega_2 T_0(-1, 0, -1)^{-1} \right] b \right\} + (0, 01) \left\{ x \left[\Omega_1 T_0(0, 1, 0)^{-1} \right] a \right. \\
 & \left. + x \left[\Omega_2 T_0(-1, 0, -1)^{-1} \right] b \right\} + (0, 0, -1) \left\{ x \left[\Omega_1 T_0(0, 1, 0)^{-1} \right] a + x \left[\Omega_2 T_0(-1, -1, 0)^{-1} \right] b \right\} + \\
 & (0, 0, -1) \left\{ x \left[\Omega_1 T_0(0, 0, 1)^{-1} \right] a + x \left[\Omega_2 T_0(0, -1, -1)^{-1} \right] b \right\} + (0, 0, 1) \left\{ x \left[\Omega_1 T_0(0, 0, 1)^{-1} \right] a + \right. \\
 & \left. x \left[\Omega_2 T_0(-1, -1, 0)^{-1} \right] b \right\} = c
 \end{aligned}$$

$\Omega_1 T_0(1, 0, 0)^{-1}, \Omega_1 T_0(0, 1, 0)^{-1}, \Omega_1 T_0(0, 0, 1)^{-1}$ is

	-1	0	1
-1	1	0	0
0	0	1	0
1	0	0	1

	-1	0	1
-1	0	0	1
0	1	0	0
1	0	1	0

	-1	0	1
-1	0	1	0
0	0	0	1
1	1	0	0

$\Omega_2 T_0(0, -1, -1)^{-1}, \Omega_2 T_0(-1, 0, -1)^{-1}, \Omega_2 T_0(-1, -1, 0)^{-1}$ is

	-1	0	1
-1	-1	0	-1
0	0	-1	0
1	-1	-1	-1

	-1	0	1
-1	0	-1	-1
0	-1	0	-1
1	-1	-1	0

	-1	0	1
-1	-1	-1	0
0	-1	-1	-1
1	0	0	-1

respec-

tively.

Step 3: Changing $\psi_1 T_0 g_{i1}^{-1}$ by $T_3 o$ to get a false function of three variables $\psi_1 T_3 o T_0 g_{i1}^{-1}$ in which variable c is a false one and Changing $\psi_2 T_0 g_{i2}^{-1}$ by $T_2 o$ to get a false function of three variables $\psi_2 T_2 o T_0 g_{i2}^{-1}$ in which variable b is a false one, respectively. Adding them to get a real function of three variables ψ_{i3} . This is the application of tension-compression operator in solving equation.

$$\psi_{i3} = \psi_1 T_3 o T_0 g_{i1}^{-1} + \psi_2 T_2 o T_0 g_{i2}^{-1} \quad (i = 1, 9)$$

$\theta_1 = \Omega_1 T_3 o T_0(1, 0, 0)^{-1} + \Omega_2 T_2 o T_0(0, -1, -1)^{-1}$ is:

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	0	-1	-1		1	0	0		0	-1	-1
0	0	1	0		-1	0	-1		0	1	0
1	-1	-1	0		-1	-1	0		-1	-1	0

$\theta_2 = \Omega_1 T_3 o T_0(1, 0, 0)^{-1} + \Omega_2 T_2 o T_0(-1, 0, -1)^{-1}$ is:

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	1	0	0		0	-1	-1		0	-1	-1
0	-1	0	-1		0	1	0		-1	0	-1
1	-1	-1	0		-1	-1	0		0	0	1

$\theta_3 = \Omega_1 T_3 o T_0(0, 1, 0)^{-1} + \Omega_2 T_2 o T_0(-1, 0, -1)^{-1}$ is:

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	0	0	1		-1	-1	0		-1	-1	0
0	0	-1	-1		1	0	0		0	-1	-1
1	-1	0	-1		-1	0	-1		0	1	0

$\theta_4 = \Omega_1 T_3 o T_0(0, 1, 0)^{-1} + \Omega_2 T_2 o T_0(-1, -1, 0)^{-1}$ is:

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	-1	-1	0		-1	-1	0		0	0	1
0	0	-1	-1		0	-1	-1		0	-1	-1
1	0	1	0		0	1	0		-1	0	-1

$\theta_5 = \Omega_1 T_3 o T_0(0, 0, 1)^{-1} + \Omega_2 T_2 o T_0(0, -1, -1)^{-1}$ is:

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	-1	0	-1		0	1	0		-1	0	-1
0	0	0	1		-1	-1	0		0	0	1
1	0	-1	-1		0	-1	-1		0	-1	-1

$\theta_6 = \Omega_1 T_3 o T_0(0, 0, 1)^{-1} + \Omega_2 T_2 o T_0(-1, -1, 0)^{-1}$ is:

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	-1	0	-1		-1	0	-1		0	1	0
0	-1	-1	0		-1	-1	0		-1	-1	0
1	1	0	0		1	0	0		0	-1	-1

Step 4: Changing ψ_{i3} by $T_0 f_i^{-1}$ we get:

$$\psi_{i4} = \psi_{i3} T_0 f_i^{-1} \quad (i = 1, 9)$$

$\theta_1 T_0(0, 0, 1)^{-1}$ is:

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	0	0	0		1	0	0		0	0	0
0	0	1	0		0	0	0		0	1	0
1	0	0	0		0	0	0		0	0	0

$\theta_2 T_0(0, 0, -1)^{-1}$ is:

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	-1	0	0		0	0	0		0	0	0
0	0	0	0		0	-1	0		0	0	0
1	0	0	0		0	0	0		0	0	-1

$\theta_3 T_0(0, 0, 1)^{-1}$ is:

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	0	0	1		0	0	0		0	0	0
0	0	0	0		1	0	0		0	0	0
1	0	0	0		0	0	0		0	1	0

$\theta_4 T_0(0, 0, -1)^{-1}$ is:

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	0	0	0		0	0	0		0	0	-1
0	0	0	0		0	0	0		0	0	0
1	0	-1	0		0	-1	0		0	0	0

$\theta_5 T_0(0, 0, -1)^{-1}$ is:

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	0	0	0		0	-1	0		0	0	0
0	0	0	-1		0	0	0		0	0	-1
1	0	0	0		0	0	0		0	0	0

$\theta_6 T_0(0, 0, 1)^{-1}$ is:

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	0	0	0		0	0	0		0	1	0
0	0	0	0		0	0	0		0	0	0
1	1	0	0		1	0	0		0	0	0

Step 5: To sum ψ_{i4} we get:

$$\psi_5 = \sum_{i=1}^9 \psi_{i4}$$

Original equation will be:

$$\psi_5(x, a, b) = c$$

$\theta_7 = \theta_1 T_0(0, 0, 1)^{-1} + \theta_2 T_0(0, 0, -1)^{-1} + \theta_3 T_0(0, 0, 1)^{-1} + \theta_4 T_0(0, 0, -1)^{-1} + \theta_5 T_0(0, 0, -1)^{-1} + \theta_6 T_0(0, 0, 1)^{-1}$ is:

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	-1	0	1		1	-1	0		0	1	-1
0	0	1	-1		1	-1	0		0	1	-1
1	1	-1	0		1	-1	0		0	1	-1

Step6: By commutation operator we get:

$$x = [C(2, 3, 0, 1)\psi_5](a, b, c) = W(a, b, c)$$

$C(2, 3, 0, 1)\theta_7$ is:

c=-1	-1	0	1	c=0	-1	0	1	c=1	-1	0	1
-1	-1	N	N		0	N	-1*0*1		1	-1*0*1	N
0	1	-1*0*1	N		-1	N	N		0	N	-1*0*1
1	0	N	-1*0*1		1	-1*0*1	N		-1	N	N

It's not oddball there are many-valued discrete points or no-valued discrete points. Not all commutation operators are used in solving equation.

Step 7: by decomposition operator we get:

$$x = \sum_{k=1}^{27} u_k \left\{ v_{k4} [v_{k1}(a) + v_{k2}(b)] + v_{k3}(c) \right\}$$

$$= \sum_{k=1}^{27} (V_k W) \left\{ (P_{k4} W) [(P_{k1} W)(a) + (P_{k2} W)(b)] + (P_{k3} W)(c) \right\}$$

we replace logogram symbols by complete ones.

$$\begin{aligned}
x = & \sum_{k=1}^{27} V_k \left[C(2, 3, 0, 1) \left(\sum_{i=1}^9 \{ [\psi_1 T_3 o T_0 (P_{i1} \psi_3)^{-1} + \psi_2 T_2 o T_0 (P_{i2} \psi_3)^{-1}] T_0 (V_i \psi_3)^{-1} \} \right) \right] \\
& \left[P_{k4} \left[C(2, 3, 0, 1) \left(\sum_{i=1}^9 \{ [\psi_1 T_3 o T_0 (P_{i1} \psi_3)^{-1} + \psi_2 T_2 o T_0 (P_{i2} \psi_3)^{-1}] T_0 (V_i \psi_3)^{-1} \} \right) \right] \right. \\
& \left(P_{k1} \left[C(2, 3, 0, 1) \left(\sum_{i=1}^9 \{ [\psi_1 T_3 o T_0 (P_{i1} \psi_3)^{-1} + \psi_2 T_2 o T_0 (P_{i2} \psi_3)^{-1}] T_0 (V_i \psi_3)^{-1} \} \right) \right] (a) \right. \\
& \left. \left. + P_{k2} \left[C(2, 3, 0, 1) \left(\sum_{i=1}^9 \{ [\psi_1 T_3 o T_0 (P_{i1} \psi_3)^{-1} + \psi_2 T_2 o T_0 (P_{i2} \psi_3)^{-1}] T_0 (V_i \psi_3)^{-1} \} \right) \right] (b) \right) \right. \\
& \left. \left. + P_{k3} \left[C(2, 3, 0, 1) \left(\sum_{i=1}^9 \{ [\psi_1 T_3 o T_0 (P_{i1} \psi_3)^{-1} + \psi_2 T_2 o T_0 (P_{i2} \psi_3)^{-1}] T_0 (V_i \psi_3)^{-1} \} \right) \right] (c) \right] \right]
\end{aligned}$$

Actually location functions $P_{ij}\psi_k$ do not change with ψ_k and can be called constant functions. Solution of equation with function Ω_1, Ω_2 and Ω_3 has been given already above by getting a function of many variables. Giving the procedure of it is just only make the method clearer. We deal with the function of three variables in solving this equation. Can we avoid to use it in the procedure? Never! In history one reason to introduce complex number is that we have to deal with complex number even if three roots of a cubic equation are all real number. It's the most important that we have gotten the solution expressed by function of one variable however.

6. COMPOSITION OF SPECIAL OPERATORS

There are 10 compositions for commutation operators and tension-compression operators and superposition operators and decomposition operators as bellow tables:

table1

	commutation	tension-compression	superposition	decomposition
commutation	1			
tension-compression	2	3		
superposition	4	5	6	
decomposition	7	8	9	10

We will mention them below. Here we give only results of binary function and they can be extended to functions of many variables easily.

Composition 1 commutation and commutation:

see table 2

Composition 2 tension-compression and commutation:

see table 3

Composition 3 tension-compression and tension-compression:

see table4

Composition 4 superposition and commutation:

is equal to commutation and superposition for commutation $C(2,1,0)$:

$$(6.1) \quad C(2,1,0)\left(\sum_{k=1}^H \psi_k\right) = \sum_{k=1}^H \left[C(2,1,0)(\psi_k)\right]$$

It will be complex for commutation $C(0,2,1)$ and commutation $C(1,0,2)$.

Composition 5 superposition and tension-compression:

is equal to tension-compression and superposition for tension-compression T_1 and T_2 :

$$(6.2a) \quad \left(\sum_{k=1}^H \psi_k\right)T_1\beta = \sum_{k=1}^H (\psi_k T_1\beta)$$

$$(6.2b) \quad \left(\sum_{k=1}^H \psi_k\right)T_2\beta = \sum_{k=1}^H (\psi_k T_2\beta)$$

is complex for tension-compression T_0 .

Composition 6 superposition and superposition:

It is very simple.

Composition 7 decomposition - commutation:

Value functions will hold the line and location functions will exchange for commutation $C(2,1,0)$

$$(6.3a) \quad V_i \left[C(2,1,0)(\psi) \right] = V_i(\psi) \quad (i = 1, L)$$

$$(6.3b) \quad P_{i1} \left[C(2,1,0)(\psi) \right] = P_{i2}(\psi) \quad (i = 1, L)$$

$$(6.3c) \quad P_{i2} \left[C(2,1,0)(\psi) \right] = P_{i1}(\psi) \quad (i = 1, L)$$

is complex for commutation $C(0,2,1)$ and commutation $C(1,0,2)$.

Composition 8 decomposition and tension-compression:

Value functions will hold the line and location functions will be acted by $T_1\beta$ or $T_2\beta$ for tension-compression T_1 and T_2 .

$$(6.4a) \quad V_i(\psi T_j\beta) = V_i\psi \quad (i = 1, L \quad j = 1, 2)$$

$$(6.4b) \quad P_{ij}(\psi T_j\beta) = (P_{ij}\psi)T_j\beta \quad (i = 1, L \quad j = 1, 2)$$

is complex for T_0 . But there is relation between value functions of ψ and of ψ acted by T_0 if it's a trivial decomposition.

$$(6.5) \quad V_i(\psi T_0\beta) = (V_i\psi)T_0\beta \quad (i = 1, L)$$

This relation is very important.

Composition 9 decomposition and superposition:

Value functions will be composition of value functions and location functions will be any location functions.

$$(6.6a) \quad V_i\left(\sum_{k=1}^H \psi_k\right) = \sum_{k=1}^H V_i(\psi_k) \quad (i = 1, L)$$

$$(6.6b) \quad P_{ij}(\sum_{k=1}^H \psi_i) = P_{ij}(\psi_k) \quad (i = 1, L \quad j = 1, 2)$$

Composition 10 decomposition and decomposition:

None.

Law and composition of special operators can be extended to high degree operators in form.

Table2 commutation and commutation

	C(1,2,0)	C(1,0,2)	C(0,2,1)	C(2,1,0)	C(2,0,1)	C(0,1,2)
C(1,2,0)	C(1,2,0)	C(1,0,2)	C(0,2,1)	C(2,1,0)	C(2,0,1)	C(0,1,2)
C(1,0,2)	C(1,0,2)	C(1,2,0)	C(2,0,1)	C(0,1,2)	C(0,2,1)	C(2,1,0)
C(0,2,1)	C(0,2,1)	C(0,1,2)	C(1,2,0)	C(2,0,1)	C(2,1,0)	C(1,0,2)
C(2,1,0)	C(2,1,0)	C(2,0,1)	C(0,1,2)	C(1,2,0)	C(1,0,2)	C(0,2,1)
C(2,0,1)	C(2,0,1)	C(2,1,0)	C(1,0,2)	C(0,2,1)	C(0,1,2)	C(1,2,0)
C(0,1,2)	C(0,1,2)	C(0,2,1)	C(2,1,0)	C(1,0,2)	C(1,2,0)	C(2,0,1)

Table3 tension-compression and commutation

	C(1,2,0)	C(1,0,2)	C(0,2,1)	C(2,1,0)	C(2,0,1)	C(0,1,2)
$T_1\beta$	$T_1\beta$	$C(1,0,2)T_1\beta$	$C(0,2,1)T_0\beta$	$C(2,1,0)T_2\beta$	$C(2,0,1)T_0\beta$	$C(0,1,2)T_2\beta$
$T_2\beta$	$T_2\beta$	$C(1,0,2)T_0\beta$	$C(0,2,1)T_2\beta$	$C(2,1,0)T_1\beta$	$C(2,0,1)T_1\beta$	$C(0,1,2)T_0\beta$
$T_0\beta$	$T_0\beta$	$C(1,0,2)T_2\beta$	$C(0,2,1)T_1\beta$	$C(2,1,0)T_0\beta$	$C(2,0,1)T_2\beta$	$C(0,1,2)T_1\beta$

table4 tension-compression and tension-compression

	$T_1\beta_2$	$T_2\beta_2$	$T_0\beta_2$
$T_1\beta_1$	$T_1(\beta_1\beta_2)$	$(T_2\beta_2)T_1\beta_1$	$(T_0\beta_2)T_1\beta_1$
$T_2\beta_1$	$(T_1\beta_2)T_2\beta_1$	$T_2(\beta_1\beta_2)$	$(T_0\beta_2)T_2\beta_1$
$T_0\beta_1$	$(T_1\beta_2)T_0\beta_1$	$(T_2\beta_2)T_0\beta_1$	$T_0(\beta_1\beta_2)$

All of them are easy to be validated by readers.

7. EXTEND RESULTS TO DISCRETE OPERATORS

Now we extend results about discrete functions to discrete operators. We limit the field of definition and range of operators within three discrete functions -e=(1,0,-1),o=(0,0,0),e=(-1,0,1) for simplicity.

Definition 7.1 Assume there are three numbers functions -e=(1,0,-1),o=(0,0,0) and e=(-1,0,1) we let $A=\{-e,0,e\}$ and define three functions operator of one variable S_1 as

$$S_1:A \rightarrow A$$

define three functions operator of two variables S_2 as

$$S_2:A^2 \rightarrow A$$

define three functions operator of three variables S_3 as

$$S_3:A^3 \rightarrow A$$

There are 3^3 single-valued three functions operators of one variable and 8^3 ones if they contain many-valued or no-valued. There are 3^9 single-valued three functions operators of two variables and 8^9 ones if they contain many-valued or no-valued. There are 3^{27} single-valued three functions operators of three variable and 8^{27} ones if they contain many-valued or no-valued.

Functions will be partitioned by the symbol '*' for many-valued point and no-valued point will be indicated by 'N'.

'+' operation will be expressed as:

	-1	0	1
-1	1	-1	0
0	-1	0	1
1	0	1	-1

'+'operator will be expressed as:

	-e	o	e
-e	e	-e	o
o	-e	o	e
e	o	e	-e

Compare two tables we know -e,o,e in discrete operators system is like -1,0,1 in discrete functions system ,respectively. We can also introduce concepts of singular three functions operator and standard singular three functions operator.

A standard singular three functions operator of two variables can be expressed by table:

	-e	o	e
-e	o	o	o
o	o	o	o
e	o	o	e

It can be represented as a superposition of three functions operators of one variable:

$$G = (o, o, e) \left[(o, o, e) \widetilde{\beta}_1 + (-e, -e, o) \widetilde{\beta}_2 \right]$$

By the same reason for three numbers function we know a standard singular binary three functions operator can be represented as a superposition of unary three functions operators and so does a general singular binary three functions operator. A general binary three functions operator can be expressed to sum of 9 singular binary three functions operators so we have

Theorem 7.1 Every binary three functions operator can be represented as a superposition of three functions operators of one variable.

A standard singular three functions operator of three variables ϕ_3 can be represented as:

$$\phi_3 = (o, o, e) \left\{ (o, o, e) \left[(o, o, e) (\widetilde{\beta}_1) + (-e, -e, o) (\widetilde{\beta}_2) \right] + (-e, -e, o) (\widetilde{\beta}_3) \right\}$$

Theorem 7.2 Every three functions operator of two or of three variables can be represented as a superposition of three functions operators of one variable.

All conclusions here are not suit to discrete 2 operator.

There are great number of operator equations constructed by 8^9 operators of two variables.

Theorem 7.3 Every operator equation constructed by three functions operators of two variables can be give formula solution represented as a superposition of three functions operators of one variable.

Although there are many operator equations we give formula solution for only double branches operator equation with digital operators and with parameterized operators.

$$(y\phi_1 f)\phi_3(y\phi_2 g) = h$$

Assume ϕ_1, ϕ_2, ϕ_3 is $\Theta_1, \Theta_2, \Theta_3$ as below, respectively:

	-e	o	e
-e	-e	e	o
o	o	-e	e
e	e	o	-e

	-e	o	e
-e	o	-e	e
o	-e	o	-e
e	e	e	o

	-e	o	e
-e	e	-e	o
o	o	e	-e
e	-e	o	e

Solution expressed by superposition of operators of one variable will be:

$$\begin{aligned}
y = & (o, o, -e) \left\{ (o, o, e) \left[(e, o, o)f + (o, -e, -e)g \right] + (o, -e, -e)h \right\} \\
& + (o, o, e) \left\{ (o, o, e) \left[(e, o, o)f + (o, -e, -e)g \right] + (-e, -e, o)h \right\} \\
& + (o, o, N) \left\{ (o, o, e) \left[(e, o, o)f + (-e, o, -e)g \right] + (o, -e, -e)h \right\} \\
& + (o, o, N) \left\{ (o, o, e) \left[(e, o, o)f + (-e, o, -e)g \right] + (-e, o, -e)h \right\} \\
& + (o, o, -e^*o^*e) \left\{ (o, o, e) \left[(e, o, o)f + (-e, o, -e)g \right] + (-e, -e, o)h \right\} \\
& + (o, o, N) \left\{ (o, o, e) \left[(e, o, o)f + (-e, -e, o)g \right] + (o, -e, -e)h \right\} \\
& + (o, o, -e^*o^*e) \left\{ (o, o, e) \left[(e, o, o)f + (-e, -e, o)g \right] + (-e, o, -e)h \right\} \\
& + (o, o, N) \left\{ (o, o, e) \left[(e, o, o)f + (-e, -e, o)g \right] + (-e, -e, o)h \right\} \\
& + (o, o, e) \left\{ (o, o, e) \left[(o, e, o)f + (o, -e, -e)g \right] + (o, -e, -e)h \right\} \\
& + (o, o, -e) \left\{ (o, o, e) \left[(o, e, o)f + (o, -e, -e)g \right] + (-e, o, -e)h \right\} \\
& + (o, o, -e^*o^*e) \left\{ (o, o, e) \left[(o, e, o)f + (-e, o, -e)g \right] + (o, -e, -e)h \right\} \\
& + (o, o, N) \left\{ (o, o, e) \left[(o, e, o)f + (-e, o, -e)g \right] + (-e, o, -e)h \right\} \\
& + (o, o, N) \left\{ (o, o, e) \left[(o, e, o)f + (-e, o, -e)g \right] + (-e, -e, o)h \right\} \\
& + (o, o, N) \left\{ (o, o, e) \left[(o, e, o)f + (-e, -e, o)g \right] + (o, -e, -e)h \right\} \\
& + (o, o, N) \left\{ (o, o, e) \left[(o, e, o)f + (-e, -e, o)g \right] + (-e, o, -e)h \right\} \\
& + (o, o, -e^*o^*e) \left\{ (o, o, e) \left[(o, e, o)f + (-e, -e, o)g \right] + (-e, -e, o)h \right\} \\
& + (o, o, e) \left\{ (o, o, e) \left[(o, o, e)f + (o, -e, -e)g \right] + (-e, o, -e)h \right\}
\end{aligned}$$

$$\begin{aligned}
& +(o, o, -e) \left\{ (o, o, e) \left[(o, o, e)f + (o, -e, -e)g \right] + (-e, -e, o)h \right\} \\
& +(o, o, N) \left\{ (o, o, e) \left[(o, o, e)f + (-e, o, -e)g \right] + (o, -e, -e)h \right\} \\
& +(o, o, -e^*o^*e) \left\{ (o, o, e) \left[(o, o, e)f + (-e, o, -e)g \right] + (-e, o, -e)h \right\} \\
& +(o, o, N) \left\{ (o, o, e) \left[(o, o, e)f + (-e, o, -e)g \right] + (-e, -e, o)h \right\} \\
& +(o, o, -e^*o^*e) \left\{ (o, o, e) \left[(o, o, e)f + (-e, -e, o)g \right] + (o, -e, -e)h \right\} \\
& +(o, o, N) \left\{ (o, o, e) \left[(o, o, e)f + (-e, -e, o)g \right] + (-e, o, -e)h \right\} \\
& +(o, o, N) \left\{ (o, o, e) \left[(o, o, e)f + (-e, -e, o)g \right] + (-e, -e, o)h \right\}
\end{aligned}$$

Definition 7.2 High Commutation Operators. Assume there is an operator of two variables ϕ , $y_1\phi y_2 = y_0$, its commutation operators $\phi(1,2,0), \phi(1,0,2), \phi(0,2,1), \phi(2,1,0), \phi(2,0,1), \phi(0,1,2)$ will be defined by following formulas and we introduce high commutation operators $\overline{C}[1,2,0], \overline{C}[1,0,2], \overline{C}[0,2,1], \overline{C}[2,1,0], \overline{C}[2,0,1], \overline{C}[0,1,2]$ then new operators can be expressed by ϕ and high commutation operators.

$$(7.1a) \quad y_1\phi[1,2,0]y_2 = y_0 \quad \phi[1,2,0] = \overline{C}[1,2,0](\phi)$$

$$(7.1b) \quad y_1\phi[1,0,2]y_0 = y_2 \quad \phi[1,0,2] = \overline{C}[1,0,2](\phi)$$

$$(7.1c) \quad y_0\phi[0,2,1]y_2 = y_1 \quad \phi[0,2,1] = \overline{C}[0,2,1](\phi)$$

$$(7.1d) \quad y_2\phi[2,1,0]y_1 = y_0 \quad \phi[2,1,0] = \overline{C}[2,1,0](\phi)$$

$$(7.1e) \quad y_2\phi[2,0,1]y_0 = y_1 \quad \phi[2,0,1] = \overline{C}[2,0,1](\phi)$$

$$(7.1f) \quad y_0\phi[0,1,2]y_1 = y_2 \quad \phi[0,1,2] = \overline{C}[0,1,2](\phi)$$

There are only two high commutation functions for a unary operator:

$$(7.2a) \quad \zeta_e(y) = y_0 \quad \zeta_e = \zeta$$

$$(7.2b) \quad \zeta_t(y_0) = y \quad \zeta_t = \overline{C}(\zeta)$$

Definition 7.3 High Tension-compression Operator. Assume there is a binary operator ϕ and an unary operator ζ , $\zeta(y_1)\phi y_2 = y_0$, we can introduce a new binary operator ϕ_1 by ϕ and ζ , ϕ_1 will meet the relation: $y_1\phi_1 y_2 = y_0$, that is say, $y_1\phi_1 y_2 = \zeta(y_1)\phi y_2$. Introduce a special high operator \overline{T}_1 to express the relation between ϕ_1 and ϕ, ζ .

$$(7.3a) \quad \phi_1 = \phi \overline{T}_1 \zeta$$

In the same way if $y_1\phi\zeta(y_2) = y_0$, we can introduce a new binary operator ϕ_2 by ϕ and ζ , ϕ_2 will meet the relation: $y_1\phi_2y_2 = y_0$, that is say, $y_1\phi_2y_2 = y_1\phi\zeta(y_2)$. Introduce a special high operator \overline{T}_2 to express the relation between ϕ_2 and ϕ, ζ .

$$(7.3b) \quad \phi_2 = \phi\overline{T}_2\zeta$$

If $y_1\phi y_2 = \zeta(y_0)$, that is say $\zeta^{-1}[y_1\phi y_2] = y_0$, we can introduce a new binary operator ϕ_0 by ϕ and ζ , ϕ_0 will meet the relation: $y_1\phi_0y_2 = y_0$, that is say $y_1\phi_0y_2 = \zeta^{-1}[y_1\phi y_2]$, and there is \overline{T}_0 :

$$(7.3c) \quad \phi_0 = \phi\overline{T}_0\zeta$$

For an unary operator we have only \overline{T} and \overline{T}_0 :

$$(7.4a) \quad \zeta_1\overline{T}\zeta_2 = \zeta_1\zeta_2$$

$$(7.4b) \quad \zeta_1\overline{T}_0\zeta_2 = \zeta_2^{-1}\zeta_1$$

Definition 7.4 High Superposition Operator. Assume there are P operators of many variables ϕ_k ($k=1, p$), its superposition operator ϕ will be:

$$(7.5) \quad \phi = \sum_{k=1}^P \phi_k$$

Function of ϕ will be the sum of function of ϕ_k ($k=1, p$). ϕ will be no-valued in a point if any of ϕ_k is no-valued in this point. $\phi_1 + \phi_2$ will be many-valued in a point if ϕ_1 is single-valued and ϕ_2 is many-valued in this point.

Definition 7.5 High Decomposition Operator.

$$(7.6) \quad \phi_3 = \sum_{i=1}^{27} \zeta_i \left\{ \eta_{i4} \left[\eta_{i1}(\widetilde{\beta}_1) + \eta_{i2}(\widetilde{\beta}_2) \right] + \eta_{i3}(\widetilde{\beta}_3) \right\}$$

We can express the relations between ζ_i or η_{ij} and ϕ_3 with special operators \overline{V}_i and \overline{P}_{ij} and actually η_{ij} is not change with ϕ_3 .

$$(7.7a) \quad \zeta_i = \overline{V}_i(\phi_3) \quad (i = 1, 27)$$

$$(7.7b) \quad \eta_{ij} = \overline{P}_{ij}(\phi_3) \quad (i = 1, 27, j = 1, 4)$$

Please note high commutation operator or high tension-compression operator or high decomposition operator or high superposition operator will be close within all three numbers operators if they contain ones being many-valued and no-valued.

Definition 7.6 False operator of M+K variables. We can change an operator of M variables to a false one of (M+K) variables by adding $\sigma\tilde{\zeta}_k$ in which σ is a zero operator and operator ϕ will not change with K variables.

$$(7.8) \quad \phi = \sum_{i=1}^L f i \sum_{j=1}^M g_{ij} \tilde{\zeta}_i = \sum_{i=1}^L f i \left\{ \sum_{j=1}^M g_{ij} \tilde{\zeta}_i + \sum_{k=M+1}^{M+K} \sigma \tilde{\zeta}_k \right\}$$

We can also get false operator of (M+K) variables from one of M variables by $\overline{T}_k\sigma$ ($k=M, M+K$).

Formula solution of double branches operator equation with parameterized operators will be:

$$\begin{aligned}
y = & \sum_{k=1}^{27} \bar{V}_k \left[\bar{C}(2, 3, 0, 1) \left(\sum_{i=1}^9 \{ [\phi_1 T_3 o \bar{T}_0 (\bar{P}_{i1} \phi_3)^{-1} + \phi_2 \bar{T}_2 \sigma \bar{T}_0 (\bar{P}_{i2} \phi_3)^{-1}] \bar{T}_0 (\bar{V}_i \phi_3)^{-1} \} \right) \right] \\
& \left[\bar{P}_{k4} \left[\bar{C}(2, 3, 0, 1) \left(\sum_{i=1}^9 \{ [\phi_1 T_3 o \bar{T}_0 (\bar{P}_{i1} \phi_3)^{-1} + \phi_2 \bar{T}_2 \sigma \bar{T}_0 (\bar{P}_{i2} \phi_3)^{-1}] \bar{T}_0 (\bar{V}_i \phi_3)^{-1} \} \right) \right] \right. \\
& \left(\bar{P}_{k1} \left[\bar{C}(2, 3, 0, 1) \left(\sum_{i=1}^9 \{ [\phi_1 T_3 o \bar{T}_0 (\bar{P}_{i1} \phi_3)^{-1} + \phi_2 \bar{T}_2 \sigma \bar{T}_0 (\bar{P}_{i2} \phi_3)^{-1}] \bar{T}_0 (\bar{V}_i \phi_3)^{-1} \} \right) \right] (f) \right. \\
& \left. \left. + \bar{P}_{k2} \left[\bar{C}(2, 3, 0, 1) \left(\sum_{i=1}^9 \{ [\phi_1 T_3 o \bar{T}_0 (\bar{P}_{i1} \phi_3)^{-1} + \phi_2 \bar{T}_2 \sigma \bar{T}_0 (\bar{P}_{i2} \phi_3)^{-1}] \bar{T}_0 (\bar{V}_i \phi_3)^{-1} \} \right) \right] (g) \right) \right. \\
& \left. \left. + \bar{P}_{k3} \left[\bar{C}(2, 3, 0, 1) \left(\sum_{i=1}^9 \{ [\phi_1 T_3 o \bar{T}_0 (\bar{P}_{i1} \phi_3)^{-1} + \phi_2 \bar{T}_2 \sigma \bar{T}_0 (\bar{P}_{i2} \phi_3)^{-1}] \bar{T}_0 (\bar{V}_i \phi_3)^{-1} \} \right) \right] (h) \right] \right]
\end{aligned}$$

Please note solution for double branches operator equation has the same form with one for double branches algebraic equation. Is it appropriate to class algebraic equation and operator equation to different fields? But we have done it! Mathematics has been parted to many alone islands. This situation is not good and will be changed in future.

These results mean that there is a new accurate analytical route beside approximate numerical method and topological way in study of operator equations certainly including functional equations and function equations and differential equations. We can extend results to N numbers operators of M variables but there are many works to be done.

8. TRY TO EXTEND TO CONTINUOUS SITUATION

We can extend results about discrete functions to continue functions if we accept results about Hilbert's 13th problem. We can express formula solution of equation constructed by continue functions in the same form of equation constructed by discrete functions even though we can't give a procedure to decompose a continue function of many variables to a superposition of functions of one variable. But there are many tasks to be done if we want to make results to be strict in logic.

We must prove that every continue operator of many variables can be represented as a superposition of continue operators of one variable if we want to extend results in this paper to continue operators and equations constructed by them. I don't know if there is such a result in current literature. Please give it if there isn't.

There are enough space for us to write our results so we are luckier than Pierre de Fermat (1601-1665) who could not write the proof of his last theorem. Now we have only poor results shown here but mathematicians will find more and more good results because there is huge mineral deposit in this direction. Please believe this point!

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